S.-T. Yau College Student Mathematics Contests 2025

Analysis and Differential Equations

Team

Please solve at least two out of the following three problems.

1. Problem 1 (This is the mathematical foundation of Artificial Intelligence)

Let f be a continuous function $[0,1]^2 \to \mathbb{R}$ such that $\|f\| = 1$, i.e. its supremum norm is one. Let t be a positive irrational number and $I = [0,1]$. Take any function $\psi_1 \in C[I] = \{I \to \mathbb{R} \text{ continuous}\}$. For any $\epsilon > 0$, there exists a N such that the variants of f on squares of width less than $1/N$ is less than $1/7$. One can construct $\phi_1 \in C[I]$ such that: $\|\psi_1 - \phi_1\| < \epsilon, \phi_1$ is constant on each of the intervals $[0/5N, 4/5N], [5/5N, 9/5N], \ldots, [1 - 5/5N, 1 - 1/5N]$ and ϕ_1 taking values of different rational numbers.

This function ϕ_1 creates a grid address system on $[0,1]^2$, divided into streets and blocks. The blocks are of form $[0/5N, 4/5N] \times [0/5N, 4/5N], [0/5N, 4/5N] \times [5/5N, 9/5N], \ldots$. To each block one can associate an unique address $\phi_1(x) + t\phi_1(y)$. Enumerate $R_{1,r}$ as the r-th block with address $a_{1,r} = \phi_1(x) + t\phi_1(y)$.

1) Show that there exists an open and dense set $U_f \subset C[I]^5$, for each $(\phi_1, ..., \phi_5) \in U_f$, associate grid address system for each ϕ_i with address $a_{i,r}$, one can construct covers of $[0,1]^2$, such that each point in $[0,1]^2$ is covered by 3 to 5 blocks and 2 to 0 streets.

2) Enumerate $R_{i,r}$ the same way as $R_{1,r}$ for each ϕ_i. For each block $R_{i,r}$, if $f > 0$ on all of $R_{i,r}$ define $g(a_{i,r}) = 1/7$; if $f < 0$ on all of $R_{i,r}$ define $g(a_{i,r}) = -1/7$. Linearly interpolate g between these defined values. Prove that $\|g\| < \frac{1.01}{7}\|f\|$, and $\|f(x,y) - \sum_{i=1}^{5} g(\phi_i(x) + t\phi_i(y))\| < \frac{6.01}{7}\|f\|$.

2. Let $\mathscr{B}$ be a Banach space, $\mathscr{V}$ a normed linear space and let L_1, L_0 be bounded linear operators from $\mathscr{B}$ to $\mathscr{V}$. For each $t \in [0,1]$, set

$$L_t = (1-t)L_0 + tL_1$$

and suppose that there is a constant C such that

$$||x||_{\mathscr{B}} \leq C||L_t x||_{\mathscr{V}}$$

for $t \in [0,1]$. Then L_1 maps $\mathscr{B}$ onto $\mathscr{V}$ if and only if L_0 maps $\mathscr{B}$ onto $\mathscr{V}$.

3. Let Ω be a bounded smooth domain and let Γ be an open, smooth portion of $\partial\Omega$. Prove that if $f, g \in C^1(\bar{\Omega})$, then the equation

$$\begin{cases} \Delta u = f, & \text{in } \Omega, \\ u = g, & \text{on } \Gamma, \\ \frac{\partial u}{\partial \vec{n}} = 0, & \text{on } \Gamma \end{cases}$$

has at most one solution.